A low Froude scheme preserving nearly-incompressible states

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Introduction

Why are we interested in geophysical flows?

- water management (quality, availability)

Introduction

Why are we interested in geophysical flows?

- forecast natural disasters, mitigate their consequences

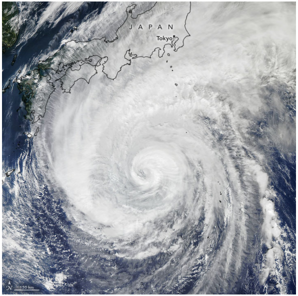
Malpasset 1959

Saint Martin Vésubie 2020

Introduction

Why are we interested in geophysical flows?

- understand interplay between ocean dynamics and
  - → the weather;
  - → climate change;
  - → the reshaping of the coastline (erosion);
  - → natural resources (marine energy, seafood);



We need a model to understand complex flow dynamics

Starting point: free surface Navier-Stokes equations

Several difficulties (discretization process):

- Conservativity and positivity of the water height;
- Keeping track of the free surface (wave rolls);
- Evolving wet/dry transitions (shore line);
- Discontinuous solutions (hydraulic jump, shock waves);

Introduction

A "simple" nonlinear model: the Shallow Water system

Reduce complexity through approximations

Assumptions:

- shallowness (characteristic depth $\ll$ domain length);
- horizontal velocity well approximated by its vertical average;
- hydrostatic pressure ($P_{\text{bottom}} = P_{\text{atm}} + g \times$ water column weight);

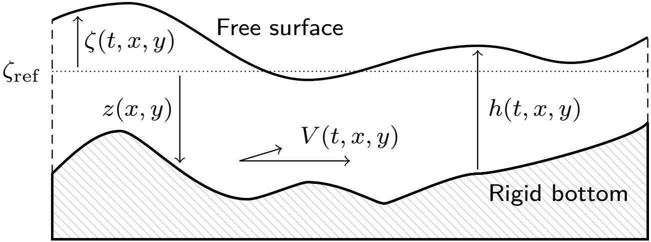
We get the Shallow Water system

Gerbeau and Perthame 2000 “Derivation of Viscous Saint-Venant System for Laminar Shallow Water; Numerical Validation”

Marche 2007 “Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects”

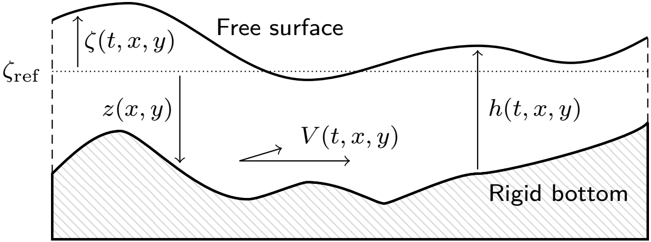
Quantities of interest:

- water height $h(t, x, y) \in \mathbb{R}_+$;
- horizontal discharge $Q(t, x, y) = (q, r)(t, x, y) \in \mathbb{R}^2$;
- horizontal velocity $V = (u, v) = Q/h \in \mathbb{R}^2$;
- bathymetry $z(x, y) \in \mathbb{R}$;



The 2D shallow water system reads:

$$\begin{cases} \dfrac{\partial h}{\partial t} + \nabla \cdot (hV) = 0 \\ \dfrac{\partial}{\partial t}(hV) + \nabla \cdot (hV \otimes V) + \nabla\left(\dfrac{g}{2}h^2\right) = -gh\nabla z \end{cases} \tag{SW}$$



The low Froude regime
Dimensionless form

Define the local Froude number $\mathrm{Fr} \stackrel{\text{def}}{=} \dfrac{|V|}{\sqrt{gh}} = \dfrac{\text{particles velocity}}{\text{acoustic waves velocity}}$

We are interested in the regime $\mathrm{Fr} \ll 1$ (multi-scale in time)

This regime is relevant:

- in coastal flows, $\mathrm{Fr} \approx 10^{-2}$;
- in river flows and lakes, $\mathrm{Fr} \approx 10^{-1}$;

What are the dominant terms in (SW) when $\mathrm{Fr} \ll 1$?
How do solutions behave?

$\longrightarrow$ Need to rewrite (SW) in *dimensionless* form

Consider the following *rescaling*:

$$\tilde{x} = \frac{x}{\ell^*}, \quad \tilde{y} = \frac{y}{\ell^*}, \quad \tilde{h} = \frac{h}{h^*}, \quad \tilde{z} = \frac{z}{h^*}, \quad \widetilde{V} = \frac{V}{v^*}, \quad \widetilde{Q} = \frac{Q}{h^* v^*}, \quad \tilde{t} = \frac{\ell^*}{v^*} t$$

System (SW) becomes:

$$\begin{cases} \dfrac{\partial \tilde{h}}{\partial \tilde{t}} + \nabla_{(\tilde{x},\tilde{y})} \cdot (\tilde{h}\widetilde{V}) = 0 \\ \dfrac{\partial}{\partial \tilde{t}}(\tilde{h}\widetilde{V}) + \nabla_{(\tilde{x},\tilde{y})} \cdot (\tilde{h}\widetilde{V} \otimes \widetilde{V}) + \dfrac{1}{\mathrm{Fr}^2} \nabla_{(\tilde{x},\tilde{y})}\left(\dfrac{\tilde{h}^2}{2}\right) = -\dfrac{\tilde{h}}{\mathrm{Fr}^2} \nabla_{(\tilde{x},\tilde{y})} \tilde{z} \end{cases} \qquad (\mathcal{P}_{\mathrm{Fr}})$$

with the characteristic Froude number

$$\mathrm{Fr} \stackrel{\text{def}}{=} v^* / \sqrt{g h^*}$$

Let $\begin{pmatrix} h \\ hV \end{pmatrix}$ be a solution of ($\mathcal{P}_{\mathrm{Fr}}$), assume it admits the asymptotic expansion:

$$\begin{aligned} h(t,x,y;\mathrm{Fr}) &= h_{(0)}(t,x,y) + \mathrm{Fr}\, h_{(1)}(t,x,y) + \mathrm{Fr}^2\, h_{(2)}(t,x,y) + \mathcal{O}(\mathrm{Fr}^3) \\ V(t,x,y;\mathrm{Fr}) &= V_{(0)}(t,x,y) + \mathrm{Fr}\, V_{(1)}(t,x,y) + \mathrm{Fr}^2\, V_{(2)}(t,x,y) + \mathcal{O}(\mathrm{Fr}^3) \end{aligned} \tag{1}$$

Plug it into ($\mathcal{P}_{\mathrm{Fr}}$) and isolate terms with same Froude powers

Consider the momentum equation

$$\frac{\partial}{\partial t}(hV) + \nabla \cdot (hV \otimes V) + \frac{1}{\mathrm{Fr}^2}\nabla\left(\frac{h^2}{2}\right) = -\frac{h}{\mathrm{Fr}^2}\nabla z$$

Extracting terms in Fr^{-2} and Fr^{-1} yields:

$$\frac{1}{\mathrm{Fr}^2}\nabla\left(\frac{h_{(0)}^2}{2}\right) = -\frac{h_{(0)}}{\mathrm{Fr}^2}\nabla z \quad \Rightarrow \quad h_{(0)}\nabla(h_{(0)} + z) = 0$$

$$\frac{1}{\mathrm{Fr}}\nabla\left(\frac{h_{(0)}h_{(1)} + h_{(1)}h_{(0)}}{2}\right) = -\frac{h_{(1)}}{\mathrm{Fr}}\nabla z \quad \Rightarrow \quad h_{(0)}\nabla h_{(1)} = 0$$

The low Froude regime

Limiting system $(\mathcal{P}_0)$

Consider the mass equation

$$\frac{\partial h}{\partial t} + \nabla \cdot (hV) = 0$$

Check terms in Fr^0 and use $\nabla(h_{(0)} + z) = 0 \Rightarrow \partial_t(h_{(0)} + z) = \partial_t h_{(0)} = \phi(t)$

$$\frac{\partial h_{(0)}}{\partial t} = -\nabla \cdot (h_{(0)} V_{(0)}) \quad \Rightarrow \quad |\Omega| \frac{\partial h_{(0)}}{\partial t} = -\int_{\partial\Omega} h_{(0)} V_{(0)} \cdot n_{|n_{\partial\Omega}} \, \mathrm{d}\sigma$$

For periodic limit conditions, the integral cancels:

$$\Omega = \mathbb{T}^2 \quad \Rightarrow \quad \frac{\partial h_{(0)}}{\partial t} = 0 \quad \Rightarrow \quad \nabla \cdot (h_{(0)} V_{(0)}) = 0$$

Back to the momentum equation

$$\frac{\partial}{\partial t}(hV) + \nabla \cdot (hV \otimes V) + \frac{1}{\mathrm{Fr}^2}\nabla\left(\frac{h^2}{2}\right) = -\frac{h}{\mathrm{Fr}^2}z$$

Terms in Fr^0 lead to:

$$\frac{\partial}{\partial t}(h_{(0)}V_{(0)}) + \nabla \cdot (h_{(0)}V_{(0)} \otimes V_{(0)}) + \nabla\Big(h_{(0)}h_{(2)} + h_{(1)}^2/2\Big) = -h_{(2)}\nabla z$$

$$\Rightarrow \frac{\partial}{\partial t}V_{(0)} + (V_{(0)} \cdot \nabla)V_{(0)} + \nabla h_{(2)} = 0$$

The low Froude regime

Limiting system $(\mathcal{P}_0)$

Define the space

$$\mathbb{W} \overset{\text{def}}{=} \{(h, V) : \mathbb{T}^2 \to \mathbb{R}^3,\ \nabla(h + z) = 0,\ \nabla \cdot (hV) = 0\} \tag{2}$$

Formally, the limiting system will write:

$$\begin{cases} \forall t \geq 0,\ (h(t, \cdot), V(t, \cdot)) \in \mathbb{W} \\ \dfrac{\partial}{\partial t} V + (V \cdot \nabla)V + \nabla \Pi = 0 \end{cases} \tag{$\mathcal{P}_0$}$$

Remark 1 (Incompressible-like space)

When $\nabla z = 0$, seeing h as a density the space $\mathbb{W}$ becomes that of incompressible states (analogy with the Euler eq.).

Remark 2 (Well prepared data)

Condition $\nabla h_{(1)} = 0$ doesn't appear in $(\mathcal{P}_0)$ but is important for $0 < \mathrm{Fr} \ll 1$.

Definition 1 (Well prepared data)

We will consider the well prepared space defined as

$$\mathbb{W}_p \overset{def}{=} \left\{ \sum_{k\in\mathbb{N}} \mathrm{Fr}^k \begin{pmatrix} h_{(k)} \\ V_{(k)} \end{pmatrix} : \mathbb{T}^2 \to \mathbb{R}^3, \begin{pmatrix} h_{(0)} \\ V_{(0)} \end{pmatrix} \in \mathbb{W}, \ \nabla h_{(1)} = 0 \right\} \quad (3)$$

- In the setting of a *flat bathymetry* ($\nabla z = 0$) and restricting to initial conditions belonging to $\mathbb{W}_p$, expansion (1) exists over some time interval $[0, T]$, $T > 0$;
- The limiting system of ($\mathcal{P}_{\mathrm{Fr}}$) has been *rigorously* shown to be ($\mathcal{P}_0$) under the previous assumptions;

Klainerman and Majda 1982 “Compressible and Incompressible Fluids”

The low Froude regime

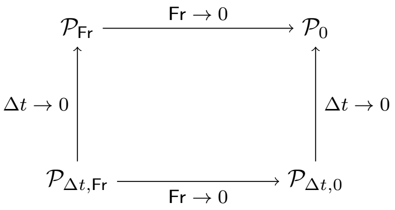
Limiting system $(\mathcal{P}_0)$

Important for a method to have the **correct asymptotic behavior**

$\rightarrow$ Consistency and stability should be *uniform* in Fr

Definition 2 (Asymptotic preserving)

$\mathcal{P}_{\Delta t,\mathrm{Fr}}$ is asymptotically consistent with $(\mathcal{P}_{\mathrm{Fr}})$ if, for all initial data, the limit scheme $\mathcal{P}_{\Delta t,0}$ results in a consistent discretization of $(\mathcal{P}_0)$. Moreover, it is asymptotically stable if the stability constraint on Δt is Fr*-independent. If both are satisfied, $\mathcal{P}_{\Delta t,\mathrm{Fr}}$ is said asymptotic preserving.*



A first IMEX scheme

Why are explicit schemes bad?

Recall: $(\mathcal{P}_{\mathrm{Fr}})$ is a system of hyperbolic conservation and balance laws

$$\frac{\partial U}{\partial t} + \nabla \cdot F(U) = S(U, z)$$

$$U = \begin{pmatrix} h \\ hV \end{pmatrix}, \quad F(U) = \begin{pmatrix} hV^T \\ hV \otimes V + h^2 \mathbf{I}_2/(2\mathrm{Fr}^2) \end{pmatrix}_{3\times 2}, \quad S(U,z) = \begin{pmatrix} 0 \\ -\frac{h}{\mathrm{Fr}^2}\nabla z \end{pmatrix}$$

Let $n \in \mathbb{S}^2$, the Jacobian $DF(U;n)$ admits the following eigenvalues:

$$\lambda_j(U;n) = (V \cdot n) + j\frac{\sqrt{h}}{\mathrm{Fr}}, \qquad j \in \{-1, 0, 1\}$$

Problem: explicit methods require prohibitively small time steps

$$\Delta t \leq \frac{\mathrm{Fr}}{2} \min\left(\frac{\Delta x}{\mathrm{Fr}\,|V \cdot n| + \sqrt{h}}\right)$$

A first IMEX scheme

Why are explicit schemes bad?

Other issues related to explicit methods:

- they are generally not asymptotically consistent;
- they make it hard to preserve lakes at rest $(K - z, 0)$

$$V = 0, \qquad \nabla\left(\frac{h^2}{2}\right) = -h\nabla z$$

In standard finite volumes schemes, the pressure is *upwinded*

$\rightarrow$ some kind of upwinding has to be enforced on the source term

Audusse et al. 2004 "A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows."

A first IMEX scheme

Wave splitting and discretization

Implicit time integration overcomes those issues...

... but it is too costly to solve nonlinear systems

Instead: try to split the system in two spatial operators:

$$\nabla \cdot F(U) - S(U, z) = H(U, z) + L(U, z)$$

$H(U, z)$ represents the convection (slow dynamics)

- its eigenvalues must remain bounded as $\mathrm{Fr} \to 0$;
- it can be nonlinear;

$L(U, z)$ represents the acoustic waves (fast dynamics)

- its eigenvalues can be unbounded as $\mathrm{Fr} \to 0$;
- it must be linear;

A first IMEX scheme

Wave splitting and discretization

Consider $(\mathcal{P}_{\mathrm{Fr}})$ in quasi-linear form

$$\frac{\partial U}{\partial t} + \begin{pmatrix} 0 & 1 & 0 \\ h/\mathrm{Fr}^2 - u^2 & 2u & 0 \\ -uv & v & u \end{pmatrix} \frac{\partial U}{\partial x} + \begin{pmatrix} 0 & 0 & 1 \\ -uv & v & u \\ h/\mathrm{Fr}^2 - v^2 & 0 & 2v \end{pmatrix} \frac{\partial U}{\partial y} = \begin{pmatrix} 0 \\ -\frac{h}{\mathrm{Fr}^2} \nabla z \end{pmatrix}$$

Chose L s.t. $\partial_t U + L(U,z) = 0$ is the linearization of $(\mathcal{P}_{\mathrm{Fr}})$ around $\begin{pmatrix} -z \\ 0 \end{pmatrix}$:

$$\frac{\partial U}{\partial t} + \begin{pmatrix} 0 & 1 & 0 \\ -z/\mathrm{Fr}^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\partial U}{\partial x} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -z/\mathrm{Fr}^2 & 0 & 0 \end{pmatrix} \frac{\partial U}{\partial y} = \begin{pmatrix} 0 \\ -\frac{h}{\mathrm{Fr}^2} \nabla z \end{pmatrix}$$

$$\Rightarrow L(U,z) = \begin{pmatrix} \nabla \cdot (hV) \\ -\frac{z}{\mathrm{Fr}^2} \nabla (h+z) \end{pmatrix}$$

A first IMEX scheme

Wave splitting and discretization

This choice of L implies

$$H(U,z) = \nabla \cdot F(U) - S(U,z) - L(U,z) = \begin{pmatrix} 0 \\ \nabla \cdot (hV \otimes V) + \frac{1}{2\mathrm{Fr}^2} \nabla (h+z)^2 \end{pmatrix}$$

- Eigenvalues of H along direction $n \in \mathbb{S}^2$ are 0, $(V \cdot n)$ and $2(V \cdot n)$;
- Eigenvalues of L are 0 and $\pm \sqrt{-z}/\mathrm{Fr}$;

Time integration will take advantage of the wave splitting

$$\frac{\partial U}{\partial t} + H(U,z) = 0 \quad \rightarrow \quad \text{explicit discretization}$$
$$\frac{\partial U}{\partial t} + L(U,z) = 0 \quad \rightarrow \quad \text{implicit discretization}$$

A first IMEX scheme

Wave splitting and discretization

Project U onto *cartesian mesh* $(C_{ij})_{N_x \times N_y}$: $\quad U_{ij} \overset{\text{def}}{=} \frac{1}{|C_{ij}|} \iint_{C_{ij}} U \, \mathrm{d}x\mathrm{d}y$

Convection: standard Rusanov flux

$$H(U_L, U_R, z_L, z_R\,; n) = \frac{1}{2}\big(H(U_L, z_L\,; n) + H(U_R, z_R\,; n) - |a|(U_R - U_L)\big) \quad (4)$$

→ Second order achieved with MUSCL reconstruction + minmod limiter

Acoustic waves: use centered differences

$$L_{ij}(U,z) = \begin{pmatrix} \frac{(hV)_{i+1,j} - (hV)_{i-1,j}}{2\Delta x} + \frac{(hV)_{i,j+1} - (hV)_{i,j-1}}{2\Delta y} \\ -\frac{z_{ij}}{\mathrm{Fr}^2} \frac{(h+z)_{i+1,j} - (h+z)_{i-1,j}}{2\Delta x} \\ -\frac{z_{ij}}{\mathrm{Fr}^2} \frac{(h+z)_{i,j+1} - (h+z)_{i,j-1}}{2\Delta y} \end{pmatrix} \quad (5)$$

Time discretization: Implicit-Explicit Runge-Kutta

$$
\begin{array}{c|ccccc}
\tilde{c}_1 & 0 & & & \\
\tilde{c}_2 & \tilde{a}_{21} & 0 & & \\
\vdots & \vdots & \vdots & \ddots & \\
\tilde{c}_s & \tilde{a}_{s1} & \tilde{a}_{s2} & \cdots & 0 \\
\hline
 & \tilde{b}_1 & \tilde{b}_2 & \cdots & \tilde{b}_s
\end{array}
\qquad
\begin{array}{c|ccccc}
c_1 & a_{11} & & & \\
c_2 & a_{21} & a_{22} & & \\
\vdots & \vdots & \vdots & \ddots & \\
c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\
\hline
 & b_1 & b_2 & \cdots & b_s
\end{array}
$$

$$
\frac{U^{(j)} - U^n}{\Delta t} + \sum_{k=1}^{j-1} \tilde{a}_{jk} H(U^{(k)}, z) + \sum_{k=1}^{j} a_{jk} L(U^{(k)}, z) = 0 \qquad \forall 1 \leq j \leq s
$$

$$
\frac{U^{n+1} - U^n}{\Delta t} + \sum_{k=1}^{s} \tilde{b}_k H(U^{(k)}, z) + \sum_{k=1}^{s} b_k L(U^{(k)}, z) = 0
$$

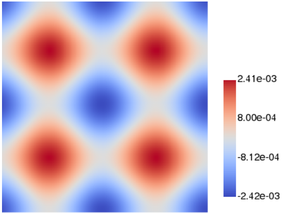
Define scheme $\mathcal{P}^1_{\Delta t,\mathrm{Fr}}$ by combining (4), (5) and the Butcher tableaux:

$$\begin{array}{c|ccc} 0 & 0 & & \\ \gamma & \gamma & 0 & \\ 1 & \delta & 1-\delta & 0 \\ \hline & \delta & 1-\delta & 0 \end{array} \qquad \begin{array}{c|ccc} 0 & 0 & & \\ \gamma & 0 & \gamma & \\ 1 & 0 & 1-\gamma & \gamma \\ \hline & 0 & 1-\gamma & \gamma \end{array} \qquad \begin{array}{l} \gamma = 1 - \sqrt{2}/2 \\ \delta = 1 - 1/(2\gamma) \end{array}$$

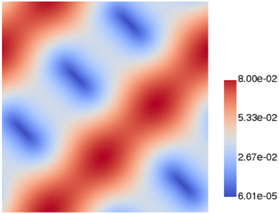
Proposition 1

Scheme $\mathcal{P}^1_{\Delta t,\mathrm{Fr}}$ is consistent at second order with $(\mathcal{P}_{\mathrm{Fr}})$ and is conservative on the water height. It is shown to preserve lakes at rest and to be asymptotically consistent. Furthermore, setting $H_{ij} \equiv 0$, its modified equation is unconditionally L^2-stable.

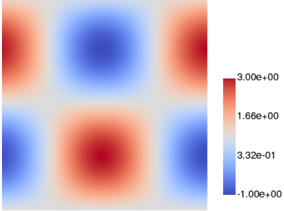
Solution at time $t = \mathrm{Fr}/6$ with $\mathrm{Fr} = 10^{-1}$ (above: reference, below: $\mathcal{P}^1_{\Delta t,\mathrm{Fr}}$)



Free surface

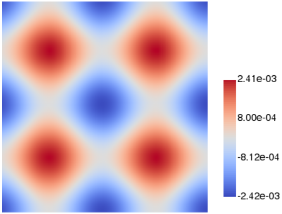


Local Froude number

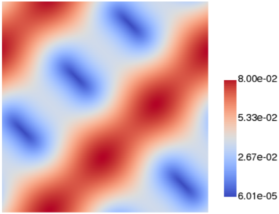


X-discharge

Solution at time $t = \mathrm{Fr}/6$ with $\mathrm{Fr} = 10^{-1}$ (above: reference, below: $\mathcal{P}^1_{\Delta t,\mathrm{Fr}}$)



Free surface



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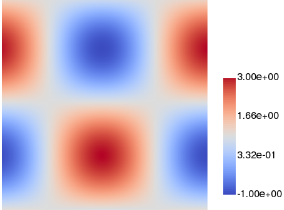

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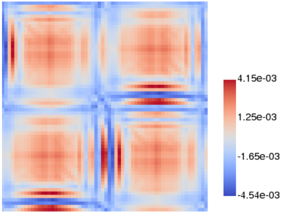
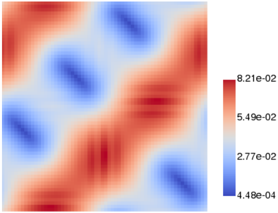


Free surface



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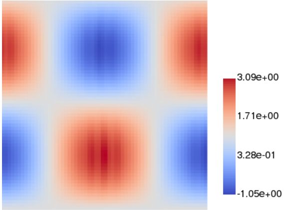
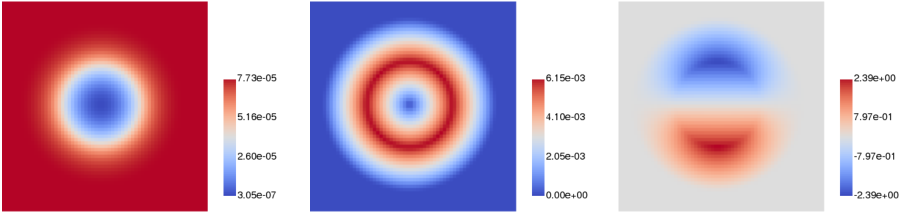


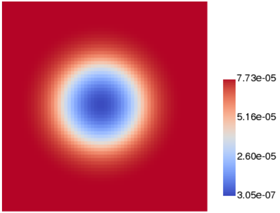
X-discharge

Gresho vortex (steady state) at time $t = 1/2$ with $\mathrm{Fr} = 10^{-2}$

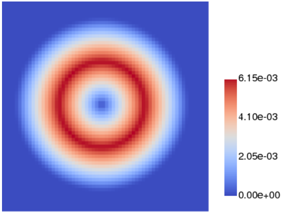


Free surface | Local Froude number | X-discharge

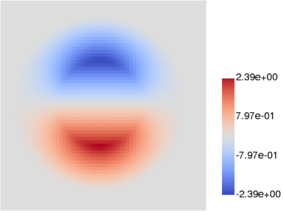
Gresho vortex (steady state) at time $t = 1/2$ with $\mathrm{Fr} = 10^{-2}$



Free surface



Local Froude number



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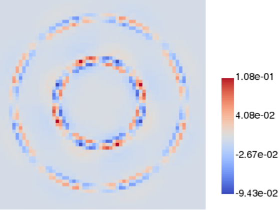

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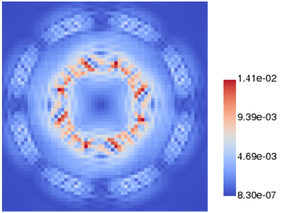

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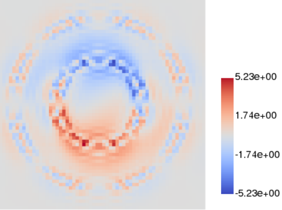


X-discharge

Recall that $\mathbb{W}_p = \left\{ \sum_{k\in\mathbb{N}} \mathrm{Fr}^k \begin{pmatrix} h_{(k)} \\ V_{(k)} \end{pmatrix} : \mathbb{T}^2 \to \mathbb{R}^3,\ \begin{pmatrix} h_{(0)} \\ V_{(0)} \end{pmatrix} \in \mathbb{W},\ \nabla h_{(1)} = 0 \right\}$

Question: Do we have $U(t=0,\cdot\,;\cdot) \in \mathbb{W}_\mathrm{p} \Rightarrow U(t>0,\cdot\,;\mathrm{Fr})$ close to $\mathbb{W}$?

Restrict to $z \equiv \mathrm{Cst}$, and introduce the L^2 spaces:

$$\mathcal{E} \overset{\text{def}}{=} \left\{ \begin{pmatrix} h \\ V \end{pmatrix} \in (L^2(\mathbb{T}^2))^3,\ \nabla h = 0,\ \nabla \cdot V = 0 \right\} = (L^2(\mathbb{T}^2))^3 \cap \mathbb{W}$$

$$\mathcal{E}^\perp \overset{\text{def}}{=} \left\{ \begin{pmatrix} h \\ V \end{pmatrix} \in (L^2(\mathbb{T}^2))^3,\ \iint_{\mathbb{T}^2} h \,\mathrm{d}x\mathrm{d}y = 0,\ \exists \phi \in H^1(\mathbb{T}^2),\ V = \nabla\phi \right\}$$

Hodge decomposition: $(L^2(\mathbb{T}^2))^3 = \mathcal{E} \oplus \mathcal{E}^\perp$ with $\mathcal{E} \perp \mathcal{E}^\perp$

$$\Rightarrow \forall U \in (L^2(\mathbb{T}^2))^3,\ \exists! \widehat{U} \in \mathcal{E},\ U - \widehat{U} \in \mathcal{E}^\perp \quad \to \text{ define } P_{\mathcal{E}} U = \widehat{U}$$

Rewrite $(\mathcal{P}_{\mathrm{Fr}})$ in (h, V) coordinates as $\partial_t U + \mathcal{K}(U) + \mathcal{G}(U) = 0$ with:

$$U = (h, V)^T, \quad \mathcal{K}(U) = (V \cdot \nabla)U, \quad \mathcal{G}(U) = (h\nabla \cdot V, \mathrm{Fr}^{-2}\nabla h)^T$$

Theorem 1 (Schochet, 1994)

Let U and U^* be respective solutions of

$$\begin{cases} \partial_t U + \mathcal{K}(U) + \mathcal{G}(U) = 0 \\ U(t = 0, \cdot) = U^0(\cdot) \end{cases} \quad (6) \qquad \begin{cases} \partial_t U^* + P_{\mathcal{E}}\mathcal{K}(U^*) = 0 \\ U^*(t = 0, \cdot) = P_{\mathcal{E}} U^0(\cdot) \end{cases} \quad (7)$$

Then $U^*(t \geq 0, \cdot) \in \mathcal{E}$, and

$$\begin{cases} \|h - \widehat{h}\|_{L^2}(t = 0) = \mathcal{O}(\mathrm{Fr}^2) \\ \|V - \widehat{V}\|_{L^2}(t = 0) = \mathcal{O}(\mathrm{Fr}) \end{cases} \Rightarrow \begin{cases} \|h - h^*\|_{L^2}(t > 0) = \mathcal{O}(\mathrm{Fr}^2) \\ \|V - V^*\|_{L^2}(t > 0) = \mathcal{O}(\mathrm{Fr}) \end{cases}$$

Preserving nearly-incompressible states

State of the art

Lin. $\mathcal{K},\mathcal{G}$ around $(-z, V^*)$: $\quad KU = (V^* \cdot \nabla)U, \quad GU = (-z\nabla \cdot V,\, \mathrm{Fr}^{-2}\nabla h)^T$

Define $\quad E_{\mathcal{E}}(t) = \mathrm{Fr}^{-2}\|\widehat{h}\|_{L^2}^2 - z\|\widehat{V}\|_{L^2}^2, \quad E_{\mathcal{E}^\perp}(t) = \mathrm{Fr}^{-2}\|h - \widehat{h}\|_{L^2}^2 - z\|V - \widehat{V}\|_{L^2}^2$

Theorem 2 (Dellacherie)

Let U and U^* be respective solutions of

$$\begin{cases} \partial_t U + KU + GU = 0 \\ U(t=0,\cdot) = U^0(\cdot) \end{cases} \qquad \begin{cases} \partial_t U^* + KU^* = 0 \\ U^*(t=0,\cdot) = P_{\mathcal{E}}U^0(\cdot) \end{cases}$$

Then $P_{\mathcal{E}}U = U^*$, and $E'_{\mathcal{E}} = E'_{\mathcal{E}^\perp} = 0$. As a consequence

$$\begin{cases} \|h - \widehat{h}\|_{L^2}(t=0) = \mathcal{O}(\mathrm{Fr}^2) \\ \|V - \widehat{V}\|_{L^2}(t=0) = \mathcal{O}(\mathrm{Fr}) \end{cases} \Rightarrow \begin{cases} \|h - h^*\|_{L^2}(t>0) = \mathcal{O}(\mathrm{Fr}^2) \\ \|V - V^*\|_{L^2}(t>0) = \mathcal{O}(\mathrm{Fr}) \end{cases}$$

Theorem 3 (Dellacherie)

Let $\mathcal{F}$ be a lin. differential operator, and let U and U^* be resp. solutions of

$$\begin{cases} \partial_t U + \mathcal{F} U = 0 \\ U(t=0,\cdot) = U^0(\cdot) \end{cases} \qquad \begin{cases} \partial_t U^* + \mathcal{F} U^* = 0 \\ U^*(t=0,\cdot) = P_{\mathcal{E}} U^0(\cdot) \end{cases}$$

The following holds:

1. $\|U^0 - P_{\mathcal{E}} U^0\|_{L^2} = \mathcal{O}(\mathrm{Fr}) \Rightarrow \|U - U^*\|_{L^2}(t \geq 0) = \mathcal{O}(\mathrm{Fr})$. *Since $\mathcal{E}$ is not invariant, in general $U^* \notin \mathcal{E}$ and thus $U^* \neq P_{\mathcal{E}} U$.*
2. *Assume $\mathcal{F}$ is such that $(\partial_t + \mathcal{F})U = 0$ leaves $\mathcal{E}$ invariant. Then we can substitute U^* with $P_{\mathcal{E}} U$ in the point above.*

$\rightarrow$ In the *linear case*, $\mathcal{E}$-invariance is sufficient to preserve nearly incompressible states

Preserving nearly-incompressible states

Low Froude accuracy

Definition 3 (Modified PDE)

The p^{th} order modified PDE associated to a scheme is an equation whose solutions are approximated by that scheme up to $\mathcal{O}(\Delta t^{p+2})$ terms.

Definition 4 (Low Froude accuracy)

Let $\mathcal{H} + \mathcal{L}$ be a wave splitting for $(\mathcal{P}_{\mathrm{Fr}})$, such that $\mathcal{H}$ has bounded eigenvalues when $\mathrm{Fr} \to 0$. A numerical scheme is low Froude accurate (LFA) for the splitting $(\mathcal{H}, \mathcal{L})$ if it admits an $\mathcal{E}$-invariant modified PDE when applied to the linearized acoustic wave equation with flat bathymetry:

$$\partial_t U + \mathcal{L}_{\text{linearized}}(U) = 0\,.$$

Arun and Samantaray 2020 “Asymptotic Preserving Low Mach Number Accurate IMEX Finite Volume Schemes for the Isentropic Euler Equations”

Preserving nearly-incompressible states

Low Froude accuracy

Proposition 2

Scheme $\mathcal{P}^1_{\Delta t,\mathrm{Fr}}$ is not low Froude accurate.

Proof. Write the modified PDE associated to $\mathcal{P}^1_{\Delta t,\mathrm{Fr}}$ when $H_{ij} = 0,\ \nabla z = 0$:

$$(\partial_t + L)U = [R_{\Delta t} - R_{\Delta x}]U \tag{8}$$

- Leading error induced by s-stages RK method $(\mathbf{A}, b)$ of order p:

$$R_{\Delta t} = \Delta t^p \Big(b^T \mathbf{A}^p \mathbf{1}_s - \frac{1}{(p+1)!} \Big) (-L)^{p+1} \quad \rightarrow \quad \text{for } \mathcal{P}^1_{\Delta t,\mathrm{Fr}},\ s = 3,\ p = 2$$

- Leading error induced by spatial centered differences of order p:

$$R_{\Delta x} = \nu_p (\Delta x^p \mathbf{L}_{n_x} \partial_x^{p+1} + \Delta y^p \mathbf{L}_{n_y} \partial_y^{p+1}) \quad \text{with} \quad \mathbf{L}_n = DF((-z,0)^T; n)$$

We have $\mathbb{W} = \ker L \subset \ker R_{\Delta t}$, but $\mathbb{W} \not\subset \ker R_{\Delta x}$ □

We would like to replace the error operator $R_{\Delta x}$ with

$$\widetilde{R}_{\Delta x} = \nu_p (\Delta x^p \partial_x^p + \Delta y^p \partial_y^p) L \quad \Rightarrow \quad \mathbb{W} \subset \ker \widetilde{R}_{\Delta x}$$

We will need to discretized operator $(\widetilde{R}_{\Delta x} - R_{\Delta x})$:

$$\widetilde{R}_{\Delta x} - R_{\Delta x} = \nu_p \Big(\Delta y^p \mathbf{L}_{n_x} \frac{\partial^{p+1}}{\partial y^p \partial x} + \Delta x^p \mathbf{L}_{n_y} \frac{\partial^{p+1}}{\partial x^p \partial y} \Big) \left(\cdot + \begin{bmatrix} z \\ 0 \end{bmatrix} \right) \tag{9}$$

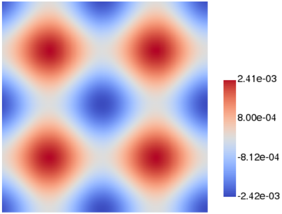
For $p = 2$, define $W = U + (z, 0)^T$ and R_{ij} a centered discretization of (9):

$$R_{ij}(U) = \frac{\nu_2}{2} \left(\frac{\mathbf{L}_{n_x}}{\Delta x} \left[W_{\cdot,j+1} - 2W_{\cdot,j} + W_{\cdot,j-1} \right]_{i-1}^{i+1} + \frac{\mathbf{L}_{n_y}}{\Delta y} \left[W_{i+1,\cdot} - 2W_{i,\cdot} + W_{i-1,\cdot} \right]_{j-1}^{j+1} \right)$$

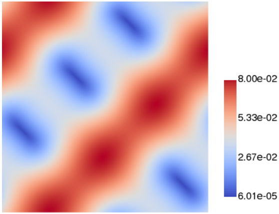
Proposition 3

Define $\mathcal{P}^2_{\Delta t,\mathrm{Fr}}$ by substituting L_{ij} with $L_{ij} + R_{ij}$ in $\mathcal{P}^1_{\Delta t,\mathrm{Fr}}$. This new scheme inherits from all the good properties of $\mathcal{P}^1_{\Delta t,\mathrm{Fr}}$, in addition of being LFA.

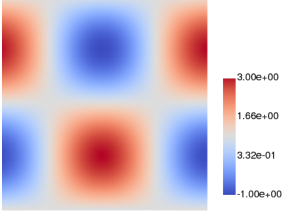
Solution at time $t = \mathrm{Fr}/6$ with $\mathrm{Fr} = 10^{-1}$ (above: reference, below: $\mathcal{P}^2_{\Delta t,\mathrm{Fr}}$)



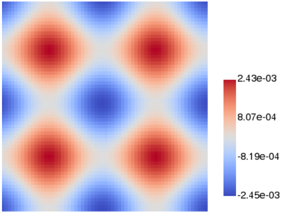
Free surface



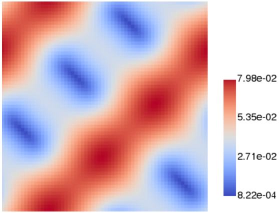
Local Froude number



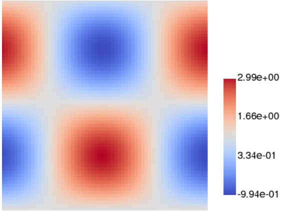
X-discharge



Free surface

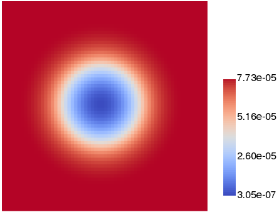


Local Froude number

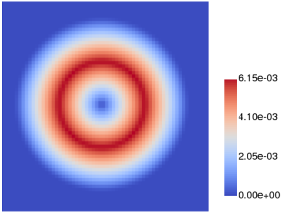


X-discharge

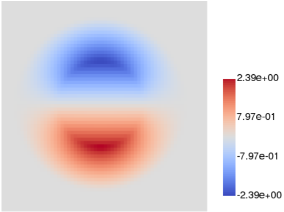
Gresho vortex (steady state) at time $t = 1/2$ with $\mathrm{Fr} = 10^{-2}$



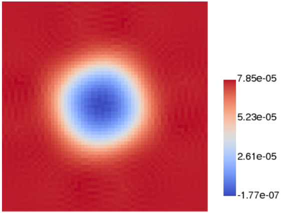
Free surface



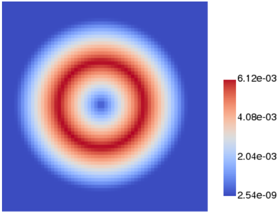
Local Froude number



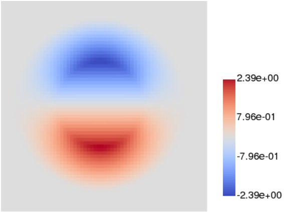
X-discharge



Free surface



Local Froude number



X-discharge

Perspectives

Conclusion: accuracy at $\mathrm{Fr} \ll 1$ requires linear $\mathcal{E}$-invariance

- second order modified scheme shows great improvement;
- procedure can be extended to higher order;

Limitations:

- high complexity due to system inversion;
- oscillations can appear in non smooth regions (MOOD procedure?);
- lack of positivity;

Ongoing work: implicit upwind kinetic scheme

- positivity and discrete entropy inequality;
- linear system can be inverted manually...
- ... but $\mathcal{O}(N^2)$ complexity;

Finite Volumes well suited for hyperbolic systems of conservation laws:

$$\frac{\partial U}{\partial t} + \nabla \cdot F(U) = 0, \quad U \in \mathbb{R}^p, \quad F(U) \in \mathbb{R}^{p\times d} \tag{10}$$

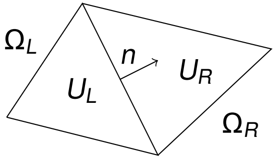
Integrate over control volume $[0, \delta t] \times \Omega$ with $\Omega \subset \mathbb{R}^d$:

$$\iint_\Omega U(\delta t, x, y) - U(0, x, y)\,\mathrm{d}x\mathrm{d}y = -\int_{\partial\Omega}\int_0^{\delta t} F(U(\tau, x, y))n_{|\partial\Omega}\,\mathrm{d}\tau\mathrm{d}\sigma$$

$$\Rightarrow \frac{\langle U\rangle_\Omega(\delta t) - \langle U\rangle_\Omega(0)}{\delta t} = -\frac{1}{|\Omega|}\int_{\partial\Omega}\langle F(U; n_{|\partial\Omega})\rangle_{[0,\delta t]}(x, y)\,\mathrm{d}\sigma \tag{11}$$

$\longrightarrow$ Need to approximate the flux at the boundary

Consider piecewise-constant data and project along the normal

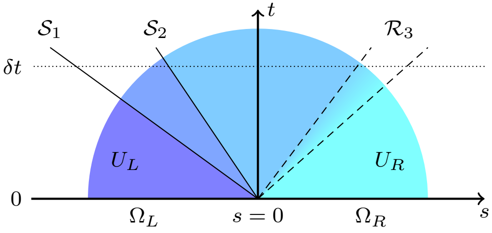


$\Rightarrow$

We get the 1D Riemann problem:

$$\begin{cases} \partial_t W + \partial_s F(W;n) = 0 \\ W^0 = \mathbf{1}_{s\leq 0} U_L + \mathbf{1}_{s>0} U_R \end{cases}$$

Self-similar solution $\widehat{W} : (t,s) \mapsto \widehat{W}(s/t; U_L, U_R)$



Hence we have $\langle F(U;n)\rangle_{[0,\delta t]} \approx F(\widehat{W}(0;U_L,U_R);n)$

Expensive to determine $\widehat{W}$ $\rightarrow$ use approximate Riemann solvers

Example: Rusanov flux

$$F_{\mathrm{Rus}}(U_L,U_R) = \frac{1}{2}(F(U_L) + F(U_R) - |a|(U_R - U_L))$$

Stability: no crossing wave (avoid collisions)
$\Rightarrow$ CFL condition on the time step

$$\Delta t \leq \frac{\Delta x}{2|a|}$$

Second order accuracy: piecwise linear reconstruction

- Edge K with neighboring cells $K|L$ and $K|R$;
- Edge normal $n_K \propto \mathrm{center}(K|R) - \mathrm{center}(K|L)$;

Reconstruction: MUSCL + minmod limiter:

$$\tilde{\nabla}_{ij} U = \left(\frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}, \frac{U_{i,j+1} - U_{i,j-1}}{2\Delta y} \right), \quad \delta_K U = \frac{U_{K|R} - U_{K|L}}{\mathrm{dist}(K|L, K|R)}$$

$$U_{K-} = U_{K|L} + \mathrm{dist}(K|L, K) \times \mathrm{minmod}(\tilde{\nabla}_{K|L} U \cdot n_K, \delta_K U)$$

$$U_{K+} = U_{K|R} - \mathrm{dist}(K, K|R) \times \mathrm{minmod}(\tilde{\nabla}_{K|R} U \cdot n_K, \delta_K U)$$

where $\quad \mathrm{minmod}(a, b) = \frac{1}{2}(\mathrm{sign}(a) + \mathrm{sign}(b)) \min(|a|, |b|)$